Avoiding Abelian Powers Cyclically

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28.9.2020

Joint work with Markus A. Whiteland

Preliminaries

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- An *abelian N-power* is a word $u_0 u_1 \cdots u_{N-1}$ if u_0, u_1, ..., u_{N-1} are abelian equivalent. For example, $010 \cdot 100 \cdot 010 \cdot 001$ is an abelian 4-power.

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  - The number N is the *exponent*.

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  - The common length $|u_0|$ is called the *period* of the abelian N-power.
  - The number N is the *exponent*.
- A word w (finite or infinite) *avoids abelian N-powers* if it contains no abelian N-power as a factor.

Classic Questions/Results

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- Thus there exists a binary word of length n avoiding abelian 4-powers for all n etc.

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- If we concatenate w with itself N times, the resulting word w^N contains at least the abelian power w^N of period $|w|$. This is unavoidable.
- But does it have to contain abelian N-powers with smaller period?
  - Maybe it does: 0010 avoids abelian 3-powers, but $(0010)^2 = 00100010$ contains the abelian 3-power 0^3 of period 1.

Main Concept

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A word w avoids abelian N-powers *cyclically* if each abelian N-power in the infinite word $www\cdots$ has period at least $|w|$.

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  - 0110 avoids abelian 3-powers, but $(0110)^\omega$ is an abelian ∞-power of period 2.
  - The converse holds (the bound is $|w|$).

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- Our generalization is thus twofold:
  - power $\rightarrow$ abelian power,
  - circular $\rightarrow$ cyclical
    - In circular avoidance w^2 is used in place of w^ω.

Main Results

Definition

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We have $5 \leq \mathcal{A}(2) \leq 8$, $3 \leq \mathcal{A}(3) \leq 4$, $2 \leq \mathcal{A}(4) \leq 3$, $\mathcal{A}(k) = 2$ *for* $k \geq 5$.

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We have $\mathcal{A}_\infty(2) = 4$, $\mathcal{A}_\infty(3) = 3$, *and* $\mathcal{A}_\infty(4) = 2$.

Values $\mathcal{A}_\infty(k)$

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A substitution $\sigma\colon A^* \to A^*$ *preserves abelian N-powers* if the following is satisfied for all words $w \in A^*$: if $\sigma(w)$ contains an abelian N-power $u_0 \cdots u_{N-1}$, then w contains an abelian N-power $v_0 \cdots v_{N-1}$ such that $\sigma(v_0 \cdots v_{N-1})$ is a conjugate of $u_0 \cdots u_{N-1}$.

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- More plainly: abelian N-powers decode to abelian N-powers up to a cyclic shift.

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- More plainly: abelian N-powers decode to abelian N-powers up to a cyclic shift.
- This is stronger than the notion of an abelian N-free substitution (we will return to this).

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Lemma

Let $\sigma \colon A^* \to A^*$ be a substitution that preserves abelian N-powers and is prolongable on the letter 0. Then the sequence $(\sigma^n(0))_n$ is a sequence of words avoiding abelian N-powers cyclically.

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Proof.

Let $z_n = \sigma^n(0)$ and $\mathbf{z}_n = z_n^\omega$ so that $\mathbf{z}_n = \sigma(\mathbf{z}_{n-1})$ for all n. Say there exists a least n such that z_n does not cyclically avoid abelian N-powers. Since $z_0 = 0$, we have $n \geq 1$.

Thus $\mathbf{z}_n$ contains an abelian N-power $u_0 \cdots u_{N-1}$ with period m, $m < |z_n|$. Since σ preserves abelian N-powers, $\mathbf{z}_{n-1}$ contains an abelian N-power $v_0 \cdots v_{N-1}$ such that $|\sigma(v_0)| = m < |z_n|$.

By the minimality of n, $|v_0| \geq |z_{n-1}|$. Hence v_0 has a conjugate z' of z_{n-1} as a factor. Therefore $m = |\sigma(v_0)| \geq |\sigma(z')| = |\sigma(z_{n-1})| = |z_n|$. $\lightning$ $\square$

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- Hence $\mathcal{A}_\infty(2) = 4$ and $\mathcal{A}_\infty(3) = 3$.
- There is no known substitution over a 4-letter alphabet that preserves abelian 2-powers.
- Thus we need something more.

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- A substitution preserving abelian N-powers is abelian N-free, but the converse is not true.
- We can prove the following.

Proposition (P.-Whiteland (2020))

Let $\sigma\colon A^* \to A^*$ be an abelian N-free substitution, and assume that $w \in A^*$ avoids abelian N-powers cyclically. If $N > 2$, then $\sigma(w)$ avoids abelian N-powers cyclically. If $N = 2$ and $|w| \geq 2$, then $\sigma(w)$ avoids abelian 2-powers cyclically.

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- Now Keränen (1992) provides us a 85-uniform substitution σ_3 (not displayed) on a 4-letter alphabet that is abelian 2-free.

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- By iterating σ_3 on 01, we see that $\mathcal{A}_\infty(4) = 2$.

Useful Lemma

Lemma

If w^ω *contains an abelian* N*-power of period* m *with* $\frac{1}{2}|w| \leq m < |w|$*, then it contains an abelian* N*-power with period* $|w| - m$.

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If w^ω contains an abelian N-power of period m with $\frac{1}{2}|w| \leq m < |w|$, then it contains an abelian N-power with period $|w| - m$.

Proof.

Assume WLOG that $m > \frac{1}{2}|w|$ and w^ω begins with an abelian N-power $u_0 \cdots u_{N-1}$ of period m. By induction on N: if w^ω begins with an abelian N-power of period m, then w^{N-1} ends with an abelian N-power $s_{N-1} \cdots s_0$ of period $|w| - m$.

Case $N = 2$. As $\frac{1}{2}|w| < m < |w|$, we have $|u_0| < |w| < |u_0 u_1|$. We may write $w = u_0 s_0$ and $u_1 = s_0 p$, where s_0 is the length $|w| - m$ suffix of w and p is a prefix of w. Notice that $|p| < m$, so we have $u_0 = p s_1$ with $|s_1| = |s_0|$. We have

$$0 = \psi(u_0) - \psi(u_1) = \psi(p s_1) - \psi(s_0 p) = \psi(s_1) - \psi(s_0).$$

Thus s_1 is abelian equivalent to s_0, and w ends with the abelian 2-power $s_1 s_0$.

Useful Lemma

Proof (Continued).

Let $N > 2$. Proceed as before and find that w ends with the abelian 2-power $s_1 s_0$ of period $|w| - m$. Conjugate w^ω to the right by $|u_0|$ to obtain z^ω that begins with the abelian $(N-1)$-power $u_1 \cdots u_{N-1}$. By the induction hypothesis, z^{N-2} ends with the abelian power $s_{N-1} \cdots s_1$ of period $|w| - m$. To conclude, we notice that $w^{N-1} = u_0 z^{N-2} s_0$. The claim follows. □

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If $w \in A^*$ avoids abelian N-powers, then $w\#$, $\# \notin A$, avoids abelian N-powers cyclically.

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Proof.

Set $\mathbf{w} = (w\#)^\omega$, and assume for a contradiction that an abelian N-power $u_0 \cdots u_{N-1}$ such that $|u_0| < |w\#|$ occurs in $\mathbf{w}$. By the previous lemma, we may assume that $|u_0| \leq \frac{1}{2}|w\#|$. Thus $|u_0 u_1| \leq |w\#|$ and $\#$ can occur in $u_0 u_1$ at most once. Thus $\#$ does not occur in u_0, and so $u_0 \cdots u_{N-1}$ must be a factor of w. This contradicts the assumption that w avoids abelian N-powers. □

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Theorem

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Proof.

Every ternary word of length 8 contains an abelian 2-power, so $\mathcal{A}(3) \geq 3$. There exists a binary word w of length n that avoids abelian 4-powers for all n (Dekking). By the previous lemma, the ternary word $w\#$ avoids abelian 4-powers cyclically. Thus $\mathcal{A}(3) \leq 4$. In the other cases, use results of Dekking and Keränen. □

Value of $\mathcal{A}(2)$

- The previous argument does not work in the binary case.
- An explicit construction is needed.

The Construction

- Let $\sigma\colon 0 \mapsto 0001, 1 \mapsto 101$ (by Dekking, the fixed point is abelian 4-free).

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Proposition

The word f avoids abelian 8-powers cyclically.

- This handles odd lengths. For even lengths, remove $\diamond$ and complement the final letter of $\overline{w}$ if $|w|$ is even.

Validity of the Construction

- Let $\mathbf{F} = f^{\omega}$.

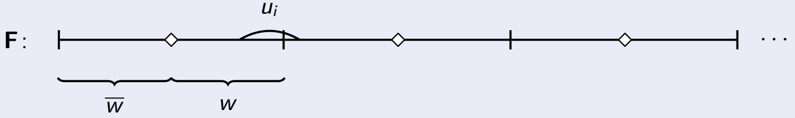
Lemma

If $\mathbf{F}$ *contains an abelian* 8*-power of period* m*, then* $m > \frac{1}{2}|w|$*.*

Validity of the Construction

Proof.

Say **F** contains abelian 8-power $u_0 \cdots u_7$. Say $m \leq \frac{1}{2}|w|$. Some u_i must "cross over" the end or middle of f; otherwise w contains an abelian 4-power.



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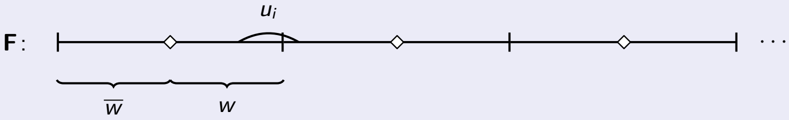
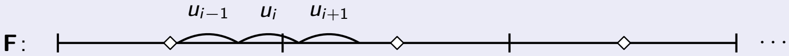


Say $1 \leq i \leq 6$, so that u_{i-1} and u_{i+1} exist. Since $m \leq \frac{1}{2}|w|$, both u_{i-1} and u_{i+1} fit inside w and $\overline{w}$.



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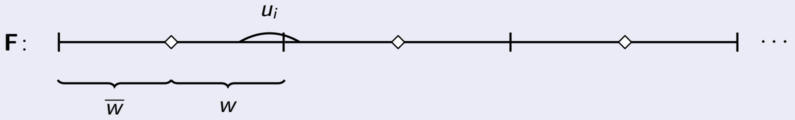

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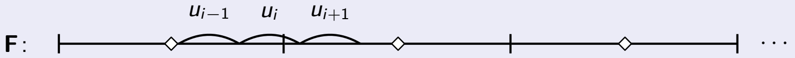

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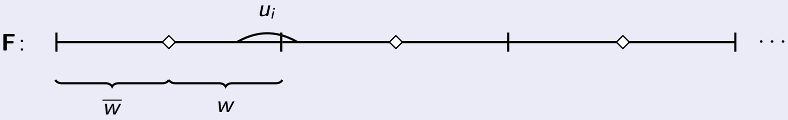

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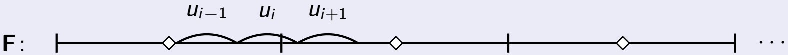

If m is large, then u_{i-1} and u_{i+1} cannot be abelian equivalent since the frequency of 0's in w is greater than that of 1's. ↯ If m is short, then abelian 4-power fits into w or $\overline{w}$ (we need the help of computer here). □

Validity of the Construction

Lemma

If **F** *contains an abelian* 8*-power* $u_0 \cdots u_7$ *of period* m *and* $m > \frac{1}{2}|w|$*, then all* u_i*'s "cross over".*

Validity of the Construction

Lemma

If **F** *contains an abelian* 8*-power* $u_0 \cdots u_7$ *of period* m *and* $m > \frac{1}{2}|w|$*, then all* u_i*'s "cross over".*

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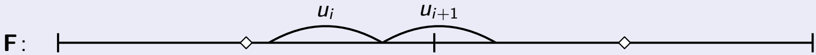

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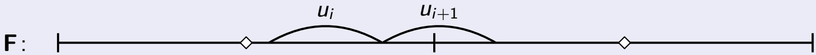
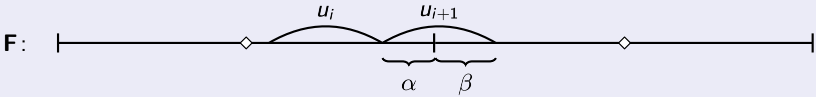


Write $u_{i+1} = \alpha\beta$ according to the "cross over" point.



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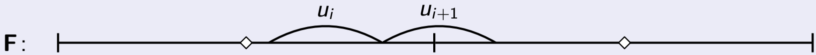

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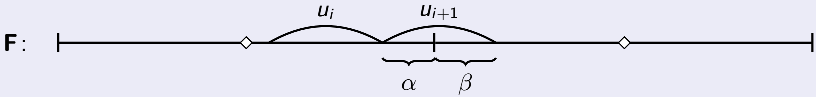

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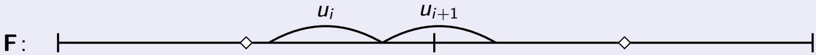

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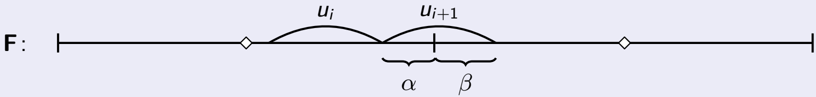

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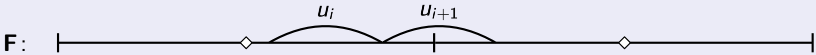

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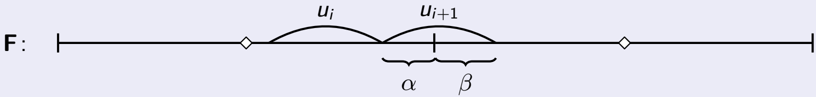

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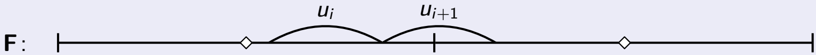

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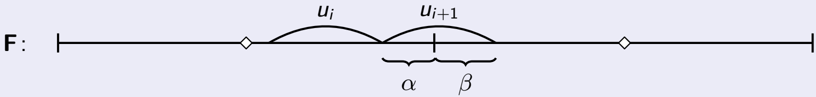

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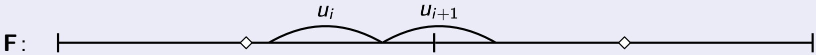

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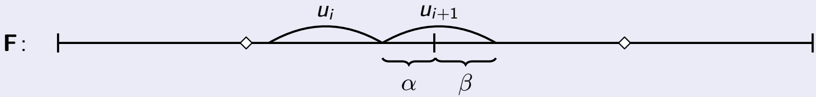

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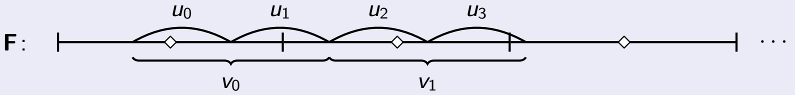

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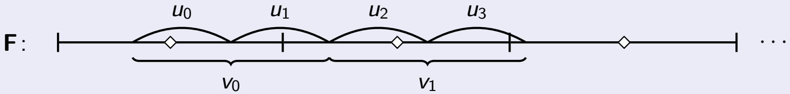

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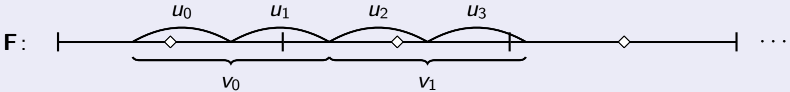

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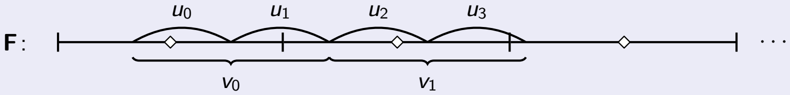

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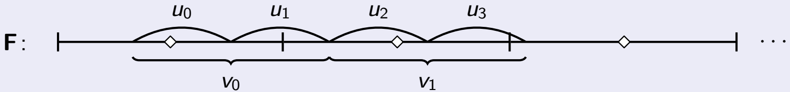

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- Since there is no binary word of length 8 avoiding abelian 4-powers cyclically, we have $\mathcal{A}(2) \geq 5$.

Open Problems

Conjecture

$\mathcal{A}(2) = 5$, $\mathcal{A}(3) = 3$, $\mathcal{A}(4) = 2$

- Verified up to length 150.

Conjecture

If $n \neq 8$, then there exists a binary word of length n avoiding abelian 4-powers cyclically.

Thank You

Thank you for your attention!

J. Peltomäki, M. A. Whiteland
Avoiding abelian powers cyclically
Adv. in Appl. Math. (to appear) (2020), arXiv:2006.06307

J. Peltomäki, M. A. Whiteland
All Growth Rates of Abelian Exponents Are Attained by Infinite Binary Words
Proceedings of MFCS 2020 (2020)