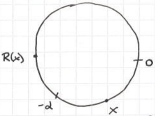
(1.)

Abelian Powers and Repetitions in Sturmian Words

Sturmian Words

- Codings of irrational rotations



- $\alpha \in [0,1]$ irrational
- $R: x \mapsto \{x+\alpha\}$
- $I_0 = [0, 1-\alpha)$, $I_1 = [1-\alpha, 1)$
- $\mathcal{L}(\alpha)$ = language of St words of slope α
- x = intercept

$s_{x,\alpha} = 10\ldots$

- Each n-letter factor $w = a_0 \cdots a_{n-1}$ corresponds to an interval $[w] = I_{a_0} \cap R^{-1}(I_{a_1}) \cap \ldots \cap R^{-(n-1)}(I_{a_{n-1}})$
- $s_{x,\alpha}$ begins with w iff $x \in [w]$

Continued Fractions

- $\alpha = [0; a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 \ddots}}$, $a_i \in \mathbb{Z}_+$

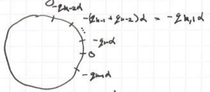
- convergents $p_k/q_k = [0; a_1, \ldots, a_k]$; they satisfy

$$p_0 = 0, \quad p_1 = 1, \quad p_k = a_k p_{k-1} + p_{k-2}$$
$$q_0 = 1, \quad q_1 = a_1, \quad q_k = a_k q_{k-1} + q_{k-2}$$

- semiconvergents $\dfrac{p_{k,\ell}}{q_{k,\ell}} = [0; a_1, \ldots, a_{k-1}, \ell] = \dfrac{\ell p_{k-1} + p_{k-2}}{\ell q_{k-1} + q_{k-2}}$

  with $1 \le \ell < a_k$

- convergents give best possible rational approximations to α; semiconvergents "second best"



Best in the sense that $\|q_k\alpha\|$ is smallest, where $\|x\| = \min\{\{x\}, 1-\{x\}\}$.

- Q_α = denominators of convergents of α
- Q_α^+ = — " — conv. & semiconvergents — " —

(2.) Powers and Periods

Thm (Damanik-Lenz)
- $w^2 \in \mathcal{L}(\alpha)$, w primitive $\Rightarrow$ $|w| \in Q_\alpha^+$,
- $|w| \in Q_\alpha^+ \setminus Q_\alpha$ $\Rightarrow$ $\exp(w) \leq 2$,
- $|w| = q_k$ $\Rightarrow$ $\exp(w) \leq a_{k+1} + 2$
- supremum of exponents $< \infty$ $\Leftrightarrow$ α has bounded partial quotients

$\overbrace{0100100}$ period 3, "covered" by power $(010)^3$

Thm (Currie-Saari) Minimal period of $w \in \mathcal{L}(\alpha)$ in Q_α^+.

Abelian Powers and Repetitions

- Goal: generalize above theorems to abelian setting.
- Joint work with Fici, Langiu, Lecroq, Lefebvre, Mignosi, Prieur-Gaston.
- TCS 2016 with the same title.
- abelian powers = powers where permutation of letters is allowed

$\overbrace{010}\cdot\overbrace{100}\cdot\overbrace{001}$, abelian power of period 3

- two words are abelian equivalent if they are permutations of each other
- w is an abelian repetition of period m if it can be covered by an abelian power of period m $\rightarrow$ generalizes period to abelian setting

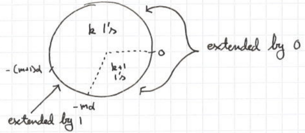
$\overbrace{01}\cdot\overbrace{010}\cdot\overbrace{100}\cdot\overbrace{001}\cdot\overbrace{1}$, abelian repetition of period 3

(3.)

- w has abelian period m if it is an abelian repetition with period m
- abelian powers harder to study in general, but relatively easy in St words thanks to the following

Thm (Fici et al.) The point $\{-m\alpha\}$ separates the intervals of factors of length m into two abelian equivalence classes. "Heavy" factors above $\{-m\alpha\}$, "light" below.

Proof sketch. By induction on m. Case $m=1$ clear. We prove the case $\{-(m+1)\alpha\} < \{-m\alpha\}$; the other case is similar.



Overall:
- below $\{-(m+1)\alpha\}$: k 1's
- between $\{-(m+1)\alpha\}$, $\{-m\alpha\}$: $k+1$ 1's
- above $\{-m\alpha\}$: $k+1$ 1's

$\square$

- Thus if $\{x\}$ and $\{x+m\alpha\}$ are on the same side of $\{-m\alpha\}$, $s_{x,\alpha}$ begins with an abelian ~~power~~ square of period m etc.
- In general: We have to see how many times $\|m\alpha\|$ fits into $|[0, -m\alpha)|$ and $|[-m\alpha, 1)|$ to find the max. exponent $ae(m)$ of period m.

Thm (F. et al) $ae(m) = \lfloor \frac{1}{\|m\alpha\|} \rfloor$.

(4.)

- If $m \in Q_\alpha^+$, then $Ae(m)$ is large. Indeed, it can be computed that $Ae(q_k) \geq q_{k+1}$.

$$f = 01\,00\,10\,10\,01\,00\,10\,10\,01\,01\ldots$$

- We can make $\|m\alpha\|$ arbitrarily small, so

Thm (Richomme–Saari–Zamboni) Sturmian words contain abelian powers of arbitrarily high exponent.

- This follows from the more general theorem

Thm (R.-S.-Z.) If an infinite word has bounded abelian complexity, then it contains abelian powers of arbitrarily high exponent.

- Since the exponent is always arbitrarily high, it does not make sense to study the max. exponent.

- Instead we measure the max. ratio between the exponent and the period of an abelian power.

abelian critical exponent: $Ac(\alpha) = \limsup_{m\to\infty} \frac{Ae(m)}{m}$

$$= \limsup_{k\to\infty} \frac{Ae(q_k)}{q_k}$$

Thm (Fici et al.) $Ac(\alpha) = \limsup_{k\to\infty} ([a_{k+1}; a_{k+2}, \ldots] + [0; a_k, a_{k-1}, \ldots, a_1])$ when $\alpha = [0; a_1, a_2, \ldots]$. Moreover, the following are equivalent:

i) $Ac(\alpha) < \infty$,
ii) $\mathcal{L}(\alpha)$ has bounded powers,
iii) α has bounded partial quotients.

- Observation: $Ac(\alpha)$ = Lagrange constant of α.

- $\mathcal{L}$ constant of α = largest c s.t. $|\alpha - \frac{p}{q}| < \frac{1}{cq^2}$ for infinitely many p/q

⑤ ## Abelian Periods

Thm (F. et al.) The min. abelian period of any factor of the Fibonacci word is a Fibonacci number.

- Proof idea: Say $F_k < m < F_{k+1}$ ($F_k = k^{th}$ Fibonacci number) and that m is an abelian period of $w \in \mathcal{L}(\alpha)$. (not necessarily min.). Then $|w| \leq (\mathrm{Ae}(m)+2)m-2$. Near the beginning of w we can always find an abelian power of period F_k and exponent $F_{k+1}+F_{k-1}-3$. This is sufficient to cover w. □

- The abelian periods are typically small:

Thm (F. et al.) Let $k \geq 3$. The min. abelian period of the finite Fibonacci word f_k is the Fibonacci number n, where

$$n = \begin{cases} \lfloor k/2 \rfloor, & \text{if } k \equiv 0,1,2 \ (4) \\ \lfloor k/2 \rfloor + 1, & \text{if } k \equiv 3 \ \ (4). \end{cases}$$

- Generalization "min. abelian period of a word in $\mathcal{L}(\alpha)$ is in Q_α^+" does not work.

- Counterexample in all other cases than Fibonacci.

- Correct formulation:

"Thm" (P.) Let m be min. abelian period of $w \in \mathcal{L}(\alpha)$. Then $m \in Q_\alpha^+$ or $m = \ell q_k$ with $1 \leq \ell \leq a_{k+1}$.

- Proof idea: Say k max. s.t. $q_k < m$. The word w must contain the singular factor of length in all phases mod q_k. Thus $|w| \geq (q_k - 1) q_{k+1} + q_k$, q_{k+1} min. return time of singular factor. On the other hand $|w| \leq (\mathrm{Ae}(m)+2)m-2$. These inequalities hold only if $\mathrm{Ae}(m)$ is large i.e. $\|m\alpha\|$ is small. Thus m corresponds to a good rational approximation of α.