All Growth Rates of Abelian Exponents Are Attained by Infinite Binary Words

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Joint work with Markus A. Whiteland

Preliminaries

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- An *abelian N-power* is a word $u_0 u_1 \cdots u_{N-1}$ if u_0, u_1, ..., u_{N-1} are abelian equivalent. For example, $010 \cdot 100 \cdot 010 \cdot 001$ is an abelian 4-power.

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  - The common length $|u_0|$ is called the *period* of the abelian N-power.
  - The number N is the *exponent*.

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- Study of abelian powers has been a hot topic in combinatorics on words recently.
- Most research concerns avoiding abelian powers, that is, finding infinite words that contain no abelian N-power for some N.
- Some important classes of infinite words (for example Sturmian words) always contain abelian powers of arbitrarily high exponent.
- To understand abelian powers better in such words, we can ask instead: how fast do the exponents grow? For a prescibed function f, does there exist an infinite word having growth rate f?

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- P.-Whiteland (2019): all growth rates θm, $\theta > 0$, are attainable, but sometimes 3 letters are required.
- Here we improve these results and show that all (reasonable) growth speeds are attained by infinite binary words.

Main result

Theorem

Let $f : \mathbb{N} \to \mathbb{R}$ be an unbounded increasing function. Then there exists an infinite binary word $\mathbf{w}$ such that the abelian exponents of $\mathbf{w}$ have growth rate f.

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- The abelian exponents of $\mathbf{w}$ grow at least as fast as f.
- It suffices to show that it does not grow faster than f.

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meaning that the abelian exponents of $\mathbf{w}$ grow at most as fast as f.

Locations for z

- Recall that $\mathbf{w} = \prod_{i=1}^{\infty} X_i$, $X_i = x_i^{\lfloor f(|x_i|) \rfloor}$, and $|x_j| \leq m < |x_{j+1}|$.

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- Recall that $\mathbf{w} = \prod_{i=1}^{\infty} X_i$, $X_i = x_i^{\lfloor f(|x_i|) \rfloor}$, and $|x_j| \leq m < |x_{j+1}|$.
- The abelian power z of period m could be located:
  - inside a block X_i,
  - on the boundary of two consecutive blocks X_i and X_{i+1},
  - or it could contain a block X_i.

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- By choosing x_i according to this theorem, we see that if z is in x_i, then $e <$ constant.
- But: Dekking's result doesn't tell us what happens if we concatenate x_i with itself.

z inside X_i, cyclic avoidance of abelian powers

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- Recall that $z = u_0 \cdots u_{e-1}$ with $|x_j| \leq m = |u_0| < |x_{j+1}|$.

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- Hence $e < \lfloor f(|x_j|) \rfloor + \text{constant}$.

Remaining cases

- What remains:
  - z on the boundary between X_i and X_{i+1} and $i < j$,
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  - z on the boundary between X_i and X_{i+1} and $i < j$,
  - z contains X_i.
- Solution:
  - Make $(|x_i|)$ grow so fast that z cannot contain X_i for $i > j$ and that the initial part $X_1 \cdots X_{j-1}$ has so small length that the exponent of z is negligible compared to $\lfloor f(|x_j|) \rfloor$ (details omitted).

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  - $\sigma(0) = 0001$,
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  - $\sigma^3(0) = 0001000100011010001000100011010001000100011011010001101$,
  - $\ldots$

Definition

A morphism σ is *abelian N-free* if $\sigma(w)$ is abelian N-free for all abelian N-free words w.

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Theorem (P.-Whiteland (2020))

Let σ be an abelian N-free morphism, and assume that w avoids abelian N-powers cyclically. If $N > 2$, then $\sigma(w)$ avoids abelian N-powers cyclically. If $N = 2$ and $|w| \geq 2$, then $\sigma(w)$ avoids abelian 2-powers cyclically.

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- Since 0 avoids abelian 4-powers cyclically, the words $\sigma(0)$, $\sigma^2(0)$, $\ldots$ avoid abelian 4-powers cyclically.
- This provides the words x_i required for the proof of the main result.

Thank You

Thank you for your attention!

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