A square root map on Sturmian words

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- Every Sturmian word can be factorized as a product of minimal squares.
- If half of each square is deleted, the resulting word is Sturmian.

Sturmian words

- $R_\alpha(x) = \{x + \alpha\}$: rotation by irrational angle $\alpha \in (0, 1)$ on the circle $[0, 1)$.
- Divide the circle: $I_0 = [0, 1 - \alpha)$, $I_1 = [1 - \alpha, 1)$.
- $s_{x,\alpha} = a_0 a_1 \cdots$, α slope, x intercept:

$$a_n = \begin{cases} 0, & \text{if } R_\alpha^n(x) \in I_0, \\ 1, & \text{if } R_\alpha^n(x) \in I_1. \end{cases}$$

- Language $\mathcal{L}(\alpha)$ independent of x.
- Every factor $w \in \mathcal{L}(\alpha)$ has a unique interval $[w]$ on the circle.
  - $s_{x,\alpha}$ begins with w if and only if $x \in [w]$.
- Assume: $\alpha < 1/2$, so $00 \in \mathcal{L}(\alpha)$ and $11 \notin \mathcal{L}(\alpha)$.

Minimal squares

- Let s be a Sturmian word of slope $[0; a+1, b+1, \ldots]$.
  - Between two blocks 1, there is 0^a or 0^{a+1}.
  - Between two blocks 10^{a+1}, there is $(10^a)^b$ or $(10^a)^{b+1}$.
- Any position in s begins with one of the six squares:

$$\begin{aligned} S_1^2 &= 0^2, & S_4^2 &= (10^a)^2, \\ S_2^2 &= (010^{a-1})^2, & S_5^2 &= (10^{a+1}(10^a)^b)^2, \\ S_3^2 &= (010^a)^2, & S_6^2 &= (10^{a+1}(10^a)^{b+1})^2, \end{aligned}$$

- The squares are *minimal*: they do not have proper square prefixes.

Minimal squares in Fibonacci

- Fibonacci word f: $a = 1$, $b = 0$.

$$\begin{aligned} S_1^2 &= 00, & S_4^2 &= 1010, \\ S_2^2 &= 0101, & S_5^2 &= 100100, \\ S_3^2 &= 010010, & S_6^2 &= 1001010010. \end{aligned}$$

- These *square roots* appear in the footer of every slide.

The square root map

- Every Sturmian word s is a product of these six minimal squares:

$$s = X_1^2 X_2^2 X_3^2 \cdots$$

Example: Fibonacci

$$\begin{aligned} f &= 0100101001001010010100100101001 0 \cdots \\ &= (010)^2 \cdot (100)^2 \cdot (10)^2 \cdot (01)^2 \cdot 0^2 \cdot (10010)^2 \cdots \end{aligned}$$

The square root map

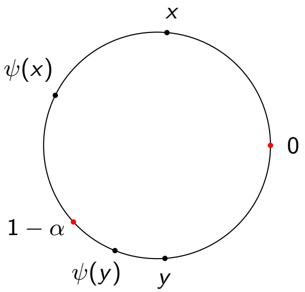
- $s = X_1^2 X_2^2 X_3^2 \cdots$
- $\sqrt{s} = X_1 X_2 X_3 \cdots$
- The square root map deletes half of each square.

Example: Fibonacci

$$f = (010)^2 \cdot (100)^2 \cdot (10)^2 \cdot (01)^2 \cdot 0^2 \cdot (10010)^2 \cdots,$$
$$\sqrt{f} = 010 \cdot 100 \cdot 10 \cdot 01 \cdot 0 \cdot 10010 \cdots$$

Special mapping ψ

- $\psi : [0,1) \to [0,1), \psi(x) = \frac{1}{2}(x + 1 - \alpha)$
- ψ halves the distance between x and $1 - \alpha$ on the interval where x belongs to.



The main result

Theorem

Let $s_{x,\alpha}$ be a Sturmian word of slope α with intercept x. Then

$$\sqrt{s_{x,\alpha}} = s_{\psi(x),\alpha}.$$

Surprisingly the square root map preserves the language of a Sturmian word.

The proof

- ψ and the minimal square roots S_i have the following key properties:

Square root property

$$\psi([S_i^2]) \subseteq [S_i]$$

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Square root property

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Shift property

$$\forall x \in [S_i^2]\colon \psi(\{x + 2|S_i|\alpha\}) = \{\psi(x) + |S_i|\alpha\}$$

Fixed points

- The only fixed point of ψ is $1-\alpha$.
- Sturmian words with intercept $1-\alpha$: $01c_\alpha$, $10c_\alpha$.

Example: Fibonacci

$$\begin{aligned} 01f &= 0101001010010010100101001001010010\cdots \\ &= (01)^2 0^2 (10)^2 (010)^2 (10010)^2 (01)^2 0^2 \cdots, \\ \sqrt{01f} &= 01 \cdot 0 \cdot 10 \cdot 010 \cdot 10010 \cdot 01 \cdot 0 \cdots = 01f. \end{aligned}$$

Non-Sturmian fixed points

- *Optimal squareful word*: aperiodic word where one of the six minimal squares begins at any position.
  - Studied and characterized by Kalle Saari.
- The square root map makes sense for these words too.
  - Sturmian words form a proper subset.

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Theorem

There exists a minimal, aperiodic subshift Ω such that for all $w \in \Omega$ either $\sqrt{w} \in \Omega$ or $\sqrt{w}$ is periodic.

The square root of an aperiodic word can be periodic!

Open problem

- There exists non-Sturmian optimal squareful words whose language is preserved under the square root map.
- Sturmian words satisfy a stronger property.
  - $\sqrt{\Omega_\alpha} \subseteq \Omega_\alpha$ for a Sturmian subshift Ω_α.

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Conjecture

Let Ω be a minimal subshift of optimal squareful words. If $\sqrt{\Omega} \subseteq \Omega$, then Ω is Sturmian.

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Thank you for your attention!