Symbolic Square Root Map

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Outline

- Introduction to the symbolic square root map

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- Introduction to a related word equation and fixed point construction

Example: Fibonacci Word

- Construct a sequence (s_k) of words as follows:

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$$s_k = s_{k-1}s_{k-2} \text{ for } k \geq 2$$

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- We have:

$$s_2 = 01$$
$$s_3 = 010$$
$$s_4 = 01001$$
$$s_5 = 01001010$$
$$\ldots$$

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- The sequence (s_n) converges to a unique limit

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- Thus $\mathbf{f}$ is expressible as a product of these squares:

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- The *square root* $\sqrt{\mathbf{f}}$ of $\mathbf{f}$ is obtained by removing half of each square:

$$\sqrt{\mathbf{f}} = 010 \cdot 100 \cdot 10 \cdot 01 \cdot 0 \cdot 100 = 01010010010100 \cdots$$

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- While $\mathbf{f} \neq \sqrt{\mathbf{f}}$, the word $\sqrt{\mathbf{f}}$ is very similar to $\mathbf{f}$.
- In fact, $\sqrt{\mathbf{f}}$ and $\mathbf{f}$ have the same language (set of factors).

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- Then $\sqrt{\cdot}$ is a continuous map $\Omega \to \Omega$ (with respect to the product topology).
- Thus $(\Omega, \sqrt{\cdot})$ is a symbolic dynamical system.

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- The limit word $\mathbf{s}$ is a *standard Sturmian word*.

Definition

An infinite word is *Sturmian* if it has the same language as some standard Sturmian word.

Generalization

- Every Sturmian word is expressible as a product of the squares of the following six words:

$$\begin{aligned} S_1 &= 0, & S_4 &= 10^{\mathfrak{a}}, \\ S_2 &= 010^{\mathfrak{a}-1}, & S_5 &= 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}}, \\ S_3 &= 010^{\mathfrak{a}}, & S_6 &= 10^{\mathfrak{a}+1}(10^{\mathfrak{a}})^{\mathfrak{b}+1} \end{aligned}$$

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- Here the integers $\mathfrak{a} \geq 1$, $\mathfrak{b} \geq 0$ depend on the integer sequence (d_k).
- Thus we may define a square root for an arbitrary Sturmian word.

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The square root map $\sqrt{\cdot}$ is a mapping $\Omega \to \Omega$.

- Proof requires linking Sturmian words to arithmetic and theory of continued fractions.

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- Can we find non-Sturmian words that have interesting behavior with respect to the square root map? Fixed points? Invariant sets?
- The rest of the talk outlines a certain fixed point construction.

Solutions to $X_1^2 \cdots X_n^2 = (X_1 \cdots X_n)^2$

- Consider a word $X_1 \cdots X_n$ that satisfies

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- Notice that 01010010 is a reversed standard word.
- If $X_1 \cdots X_n$ is a solution, then $X_1 \cdots X_n$ is a prefix of $X_1^2 \cdots X_n^2$.
- Thus if $\mathbf{s}$ has infinitely many prefixes $X_1^2 \cdots X_n^2$ that satisfy (1), then $\sqrt{\mathbf{s}} = \mathbf{s}$.

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Reversed standard words are solutions to $X_1^2 \cdots X_n^2 = (X_1 \cdots X_n)^2$.

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- Reversed standard words give all Sturmian fixed points.
- What other solutions do we have?

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Definition

Let $u = a_0 a_1 \cdots a_{n-1}$, $a_i \in \{S, L\}$ be a word over $\{S, L\}$. The word u is a *pattern word* if $a_i = a_j$ whenever i and j are in the same orbit of the map $x \mapsto 2x \pmod{n}$.

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Let w be a solution to $X_1^2 \cdots X_n^2 = (X_1 \cdots X_n)^2$ and w' be the word obtained from w by exchanging its first two letters. If u is a pattern word, then we denote by $\mathcal{P}_w(u)$ the word obtained from u by replacing S by w and L by w'.

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- If $w = 01010010$, then $w' = 10010010$. We have $\mathcal{P}_w(LSS) = 10010010 \cdot 01010010 \cdot 01010010$.

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Theorem (P.-Saarela-Whiteland (2014,2019))

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The solutions to $X_1^2 \cdots X_n^2 = (X_1 \cdots X_n)^2$ are exactly

1. *reversed standard words,*
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- Thus we get two classes of fixed points corresponding to the two solution types.

Open Problems

- What other fixed points exist?
- What other $\sqrt{\cdot}$-invariant sets exist?